Combinatorial Networks
May 20th-21st,Thursday

- **Thm1(Strong Perfect Graph Theorem):**

  G is perfect iff G has no odd hole nor the complement of odd hole.

  Remark "odd hole" is the induced odd cycle of length ≤ 5.

  Given G, define $A = A(G)$ as follow:

$$A_{ij} = \begin{cases} 0, & ij \notin E(G), \\ x_{ij}, & ij \in E(G) \text{ and } i < j, \\ -x_{ji}, & ij \in E(G) \text{ and } i > j, \end{cases}$$

- **Thm2(Tultes/Lovàsz):** G has a perfect matching iff $Det(A) \not\equiv 0$.

  Proof. Recall that

$$Det(A) = \sum_{\pi} sign(\pi) \prod_{i=1}^{n} A_{i\pi(i)}.$$

  For $\forall\ \pi$, $E_\pi \overset{\text{Def}}{=} \{(i, \pi(i)) \in E(G), \forall\ i\}$. So

$$Det(A) = \sum_{\substack{\pi \\ E_\pi \subseteq E(G)}} sign(\pi) \prod_{i=1}^{n} A_{i\pi(i)}.$$

  Note that E_π is a collection of cycle in G.

  Let $\overrightarrow{E_\pi} = \{\overrightarrow{(i, E(i))} : \forall\ i\}$. Then $\overrightarrow{E_\pi}$ is a collection of directed cycles.

  Observation: G has a PM iff there exits π such that $\overrightarrow{E_\pi}$ is a collection of even directed cycles.

- **Claim1:** If G has a PM, then $Det(A) \not\equiv 0$.

  Proof. Let $(i, t(i))$ be a PM in G, and consider the permutation $\pi : i \mapsto t(i), t(i) \mapsto i$. Then

$$\prod_{i=1}^{n} A_{i\pi(i)} = (-1)^{\frac{n}{2}} \prod_{(i,t(i)) \in M} x_{it(i)}^2.$$

  This type of monomials cannot be cancelled by others. So $Det(A) \not\equiv 0$.

- **Claim2:** If G has no PM, then $Det(A) \equiv 0$.

  Proof. By induction, for $\forall\ \pi$ for which $\prod_{i=1}^{n} A_{i\pi(i)} \not\equiv 0, \overrightarrow{E_\pi}$ has an odd directed cycle. Let $\overrightarrow{C_\pi}$ be the odd cycle in $\overrightarrow{E_\pi}$ which contains the "smallest" vertex. We will pair those non-zero monomials $\prod A_{i\pi(i)}$ in a way that one cancels the other.

  For each $\overrightarrow{E_\pi}$, let $f(\overrightarrow{E_\pi})$ be the graph obtained from $\overrightarrow{E_\pi}$ by reversing the direction of the arcs in $\overrightarrow{C_\pi}$, then $f(f(\overrightarrow{E_\pi})) = \overrightarrow{E_\pi}$. So we have a pairing of $\{\overrightarrow{E_\pi} : \pi\}$.

  Let $\overrightarrow{E_{\pi'}} = f(\overrightarrow{E_\pi})$. Then

(1) $sign(\pi) = sign(\pi')$;

(2) $\prod A_{i\pi'(i)} = (-1)^{|V(C_\pi)|} \prod A_{i\pi(i)} = (-1) \prod A_{i\pi(i)}$.

So $Det(A) \equiv 0$.

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- **Thm3:** $ex(n, C_{2t}) = O(tn^{1+\frac{1}{t}})$.

  Proof. Suppose not, then there exits $C_{2t}-$free graph G with $\Omega(tn^{1+\frac{1}{t}})$ edges. So there is $G' \subset G$, which is bipartite and $\delta(G') = \Omega(tn^{\frac{1}{t}})$. Consider G' and the BFS-tree:

  Fix $v \in V(G')$, define:

  $L_0 = \{v\}, L_1 = N(V)$;

  $L_{i+1} = \{w \in V(G') \setminus \bigcup_{j=0}^{i} L_j : \exists\, u \in L_i, uw \in E(G')\} = N(L_i) \setminus \bigcup_{j=0}^{i} L_j,\ i = 1, 2, \cdots, t-1.$

- **Claim1:** For $1 \le i \le t,\ e(L_{i-1} \cup L_i) \le 4t(|L_{i-1}| + |L_i|)$.

  Proof. Suppose not, then $G'(L_{i-1} \cup L_i)$ has a bipartite G_0 with $\delta(G_0) \ge 4t$. So G_0 has a cycle C of length $\ge 4t$, which is even. Assume $C = x_1\text{-}x_2 \cdots\text{-}x_{4t} \cdots\text{-}x_{2s}\text{-}x_1$.

  Lemma: Let C be a cycle with an arc and $V(C) = X \cup Y$ $(X \cap Y = \emptyset)$, then for $2 \le 2i \le \frac{1}{2}|C|$, there exits an X-Y path of length $2i$. (X-Y path is a path of C from $x \in X$ to $y \in Y$)

  Let $T' \subset T$ be the minimal subtree containing all $L_{i-1} \cap C$. Since T' is minimal, T' has at least 2 branches, one of which is B_1.

  Let $X = B_1 \cap C,\ Y = C \setminus X.\ \forall\, a \in X, b \in Y \cap L_{i-1}, \exists\ a$-$b$ path of length $2h \le 2(t-1)$.(Assume h is the height of T') Pick $2i$ such that $2i+2h = 2t$. By lemma, there exits X-Y path P in C of length $2i$ from x to y. Since P is of even, $x \in X$ implies that $y \in Y \cap L_{i-1} = T' \cap L_{i-1} - B_1$. But in $T', \exists\ x$-y path Q of length $2h$. So $P \cup Q$ is a cycle of length $2i + 2h = 2t$. A contradiction!

- **Claim2:** $|L_{i+1}| \ge |L_i|\Omega(n^{\frac{1}{t}})$

  Proof. We have:

  (1) $e(L_{i-1} \cup L_i) + e(L_i \cup L_{i-1}) = \sum_{v \in L_i} d_{G'}(v) \ge |L_i|\Omega(tn^{\frac{1}{t}})$;

  (2) $e(L_{i-1} \cup L_i) \le 4t(|L_{i-1}| + |L_i|)$;

  (3) $e(L_i \cup L_{i+1}) \le 4t(|L_i| + |L_{i+1}|)$.

  So $4t(|L_{i-1}|+2|L_i|+|L_{i+1}|) \ge |L_i|\Omega(tn^{\frac{1}{t}}) \Rightarrow |L_{i-1}|+2|L_i|+|L_{i+1}| \ge |L_i|\Omega(tn^{\frac{1}{t}}) \Rightarrow |L_{i+1}| \ge |L_i|\Omega(tn^{\frac{1}{t}})$.

  It's done as $|L(t)| = \Omega(n)$. ∎